

An exact solution of vacuum field equation in Finsler spacetime

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Abstract

The vacuum field equation in Finsler spacetime is equivalent to vanishing of Ricci scalar. We present an exact solution of the Finslerian vacuum field equation. The solution is similar to the Schwarzschild metric. It reduces to Schwarzschild metric while the Finslerian parameter vanishes. We get solutions of geodesic equation in such a Schwarzschild-like spacetime, and show that the geodesic equation returns to the counterpart in Newton's gravity at weak field approximation. It is proved that the Finslerian covariant derivative of the geometrical part of the gravitational field equation is conserved. The interior solution is also given.

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I. INTRODUCTION

In 1912, Einstein proposed his famous general relativity which gives the connection between Riemann geometry and gravitation. In general relativity, the effects of gravitation are ascribed to spacetime curvature instead of a force. In four dimensional spacetime, two solutions of the Einstein vacuum field equation are well known [1]. These are the Schwarzschild solution which preserves spherical symmetry, and the Kerr solution which preserves axial symmetry. The Schwarzschild solution is of vital importance for general relativity. The physics of Schwarzschild solution is quite different from Newton's gravity. The success of general relativity attribute to the four classical tests [2]. The predictions of the four classical tests directly come from the Schwarzschild solution. Most celestial bodies can be approximately treated as a sphere. Thus, the Schwarzschild solution is widely used in investigating the astronomical phenomena. However, the recent astronomical observations show that the gravitational field of galaxy clusters offset from its baryonic matters [3]. It implies that the spherical symmetry may not preserve in the scale of galaxy clusters.

Finsler geometry [4] is a new geometry which involves Riemann geometry as its special case. S. S. Chern pointed out that Finsler geometry is just Riemann geometry without quadratic restriction, in his Notices of AMS. The symmetry of spacetime is described by the isometric group. The generators of isometric group are directly connected with the Killing vectors. It is well known that the isometric group is a Lie group in Riemannian manifold. This fact also holds in Finslerian manifold [5]. Generally, Finsler spacetime admits less Killing vectors than Riemann spacetime does [6]. The numbers of independent Killing vectors of an n dimensional non-Riemannian Finsler spacetime should no more than $\frac{n(n-1)}{2}+1$ [7]. The causal structure of Finsler spacetime is determined by the vanish of Finslerian length [8]. And the speed of light is modified. It has been shown that the translation symmetry is preserved in flat Finsler spacetime [6]. Thus, the energy and momentum are well defined in Finsler spacetime. In flat Finsler spacetime, inertial motion preserves the Finslerian length and admits a modified dispersion relation.

Gibbons *et al.* [9] have pointed out that Glashow's Very Special Relativity [10] is Finsler Geometry. The flat Finsler spacetime breaks the Lorentz symmetry. Thus, it is a possible mechanism of Lorentz violation [11]. Stavrinou *et al.* [12] used the method of osculating Riemannian space to study the cosmological anisotropy in Finsler spacetime. Vacaru *et al.*

studied high dimensional gravity in Finsler spacetime [13].

The counterpart of special relativity has been established in flat Finsler spacetime. However, up to now, Finslerian gravity is still to be completed. There are types of gravitational field equation in Finsler spacetime [14–18]. And these equations are not equivalent to each other. Here are the reasons. It is well known that there is only a torsion free connection—the Christoffel connection in Riemann geometry. However, there are types of connection in Finsler geometry. Therefore, the covariant derivatives that depend on the connection are different. The Finslerian length element F is constructed on a tangent bundle [4]. Thus, the gravitational field equation should be constructed on the tangent bundle in principle. However, the corresponded energy-momentum tensor, which should be constructed on the tangent bundle, is rather obscure in physics.

The analogy between geodesic deviation equations in Finsler spacetime and Riemann spacetime gives the vacuum field equation in Finsler gravity [19]. It is the vanishing of Ricci scalar. The vanishing of the Ricci scalar implies that the geodesic rays are parallel to each other. The geometric invariant property of Ricci scalar implies that the vacuum field equation is insensitive to the connection, which is an essential physical requirement. In this paper, we present an exact solution of the vacuum field equation in Finsler spacetime. The interior solution is also shown.

This paper is organized as follows. Sec.II is divided into three subsections. In subsection A, we briefly introduce some basic geometric objects in Finsler geometry. In subsection B, we present an exact solution of the vacuum field equation in Finsler spacetime. In subsection C, we investigate the Newtonian limit of our solution. In Sec. III, we propose a gravitational field equation with source. We prove that the Finslerian covariant derivative of the geometrical part of the gravitational field equation is conserved. Then, an interior solution of gravitational field equation is shown. In Sec. IV, we investigate particle motion in the Schwarzschild-like spacetime that is given in Sec. II. We get three solutions of geodesic equations. In Sec. V, we present an example of two dimensional Finsler space with constant positive flag curvature, which is a subspace of the Schwarzschild-like spacetime. Then, we discuss the geometric properties of the two dimensional Finsler space. Conclusions and remarks are given in Sec. VI.

II. THE EXACT SOLUTION OF VACUUM FIELD EQUATION

A. Formulism of Finsler geometry

Instead of defining an inner product structure over the tangent bundle in Riemann geometry, Finsler geometry is based on the so called Finsler structure F with the property $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$, where $x \in M$ represents position and $y \equiv \frac{dx}{d\tau}$ represents velocity. The Finslerian metric is given as [4]

$$g_{\mu\nu} \equiv \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y^\nu} \left(\frac{1}{2} F^2 \right). \quad (1)$$

A Finslerian metric is said to be Riemannian, if F^2 is quadratic in y . Throughout this paper, the indices are lowered and raised by $g_{\mu\nu}$ and its inverse matrix $g^{\mu\nu}$.

The geodesic equation for Finsler manifold is given as

$$\frac{d^2 x^\mu}{d\tau^2} + 2G^\mu = 0, \quad (2)$$

where

$$G^\mu = \frac{1}{4} g^{\mu\nu} \left(\frac{\partial^2 F^2}{\partial x^\lambda \partial y^\nu} y^\lambda - \frac{\partial F^2}{\partial x^\nu} \right) \quad (3)$$

is called geodesic spray coefficients. It can be proved from the geodesic equation (2) that the Finslerian structure F is constant along the geodesic.

In Finsler geometry, there is geometrical invariant quantity, i.e., Ricci scalar. It is of the form

$$Ric \equiv R^\mu_\mu = \frac{1}{F^2} \left(2 \frac{\partial G^\mu}{\partial x^\mu} - y^\lambda \frac{\partial^2 G^\mu}{\partial x^\lambda \partial y^\mu} + 2G^\lambda \frac{\partial^2 G^\mu}{\partial y^\lambda \partial y^\mu} - \frac{\partial G^\mu}{\partial y^\lambda} \frac{\partial G^\lambda}{\partial y^\mu} \right), \quad (4)$$

where $R^\mu_\nu = R^\mu_{\lambda\nu\rho} y^\lambda y^\rho / F^2$. Though $R^\mu_{\lambda\nu\rho}$ depends on connections, R^μ_ν does not [4]. The Ricci scalar only depends on the Finsler structure F and is insensitive to connections. The analogy between geodesic deviation equations in Finsler spacetime and Riemann spacetime gives the vacuum field equation in Finsler gravity [19]. It is of the form $Ric = 0$.

B. Vacuum solution

Here, we propose an ansatz that the Finsler structure is of the form

$$F^2 = B(r) y^t y^t - A(r) y^r y^r - r^2 \bar{F}^2(\theta, \varphi, y^\theta, y^\varphi). \quad (5)$$

Then, the Finsler metric can be derived as

$$g_{\mu\nu} = \text{diag}(B, -A, -r^2 \bar{g}_{ij}), \quad (6)$$

$$g^{\mu\nu} = \text{diag}(B^{-1}, -A^{-1}, -r^{-2} \bar{g}^{ij}), \quad (7)$$

where \bar{g}_{ij} and its reverse are the metric that derived from \bar{F} and the index i, j run over angular coordinate θ, φ .

Plugging the Finsler structure (5) into the formula (3), we obtain that

$$G^t = \frac{B'}{2A} y^t y^r, \quad (8)$$

$$G^r = \frac{A'}{4A} y^r y^r + \frac{B'}{4A} y^t y^t - \frac{r}{2A} \bar{F}^2, \quad (9)$$

$$G^\theta = \frac{1}{r} y^\theta y^r + \bar{G}^\theta, \quad (10)$$

$$G^\varphi = \frac{1}{r} y^\varphi y^r + \bar{G}^\varphi, \quad (11)$$

where the prime denotes the derivative with respect to r , and the \bar{G} is the geodesic spray coefficients derived by \bar{F} . Plugging the geodesic coefficients (8,9,10,11) into the formula of Ricci scalar (4), we obtain that

$$\begin{aligned} F^2 Ric = & \left[\frac{B''}{2A} - \frac{B'}{4A} \left(\frac{A'}{A} + \frac{B'}{B} \right) + \frac{B'}{rA} \right] y^t y^t \\ & + \left[-\frac{B''}{2B} + \frac{B'}{4B} \left(\frac{A'}{A} + \frac{B'}{B} \right) + \frac{A'}{rA} \right] y^r y^r \\ & + \left[\bar{Ric} - \frac{1}{A} + \frac{r}{2A} \left(\frac{A'}{A} - \frac{B'}{B} \right) \right] \bar{F}^2, \end{aligned} \quad (12)$$

where \bar{Ric} denotes the Ricci scalar of Finsler structure \bar{F} . Since \bar{F} is independent of y^t and y^r , the vanish of Ricci scalar implies that the term in each square bracket of equation (12) should vanish respectively. These equations are given as

$$0 = \frac{B''}{2A} - \frac{B'}{4A} \left(\frac{A'}{A} + \frac{B'}{B} \right) + \frac{B'}{rA}, \quad (13)$$

$$0 = -\frac{B''}{2B} + \frac{B'}{4B} \left(\frac{A'}{A} + \frac{B'}{B} \right) + \frac{A'}{rA}, \quad (14)$$

$$0 = \bar{Ric} - \frac{1}{A} + \frac{r}{2A} \left(\frac{A'}{A} - \frac{B'}{B} \right). \quad (15)$$

Noticing that \bar{Ric} is independent of r , thus the equation (15) is satisfied if and only if \bar{Ric} equals to constant. It means that the two dimensional Finsler space \bar{F} is a constant flag

curvature space. The flag curvature is a generalization of sectional curvature in Riemann geometry. Here, we label the constant flag curvature to be λ . Therefore, $\bar{Ric} = \lambda$. The equations (13,14,15) are similar to the Schwarzschild vacuum field equation in general relativity. The solutions of equations (13,14,15) are given as

$$B = a\lambda + \frac{b}{r}, \quad (16)$$

$$A = \left(\lambda + \frac{b}{ra} \right)^{-1}, \quad (17)$$

where a and b are integral constants.

C. The Newtonian limit

In the above subsection, we have obtained the vacuum field solution in Finsler spacetime. The integral constants of solutions (16,17) should be determined by specific boundary conditions which is given by physical requirement. Here, we require that the solutions should return to Newton's gravity in weak field approximation [2]. In order to compare with Newton's gravity, we only consider the radial motion of particles. Plugging the solutions (16,17) into the geodesic coefficients (8,9), and noticing that the velocity of particle $\frac{dr}{dt}$ is small, we obtain the geodesic equations

$$\frac{d^2 t}{d\tau^2} = 0, \quad (18)$$

$$\frac{d^2 r}{d\tau^2} - \frac{b\lambda}{2r^2} \left(\frac{dt}{d\tau} \right)^2 = 0. \quad (19)$$

Combining the geodesic equations (18,19), we obtain that

$$\frac{d^2 r}{dt^2} = \frac{b\lambda}{2r^2}. \quad (20)$$

Comparing the equation (20) with Newton's gravity, we conclude that

$$b\lambda = -2GM, \quad (21)$$

where M denotes the total mass of gravitational source.

III. INTERIOR SOLUTION

It is well known that the Schwarzschild spacetime has interior solution, which can deduce the famous Oppenheimer-Volkoff equation. The interior behavior of Finsler spacetime (5) is

worth investigating. However, as we mentioned in the introduction, there are obstructions in constructing gravitational field equation in Finsler spacetime. In this section, we will show that there is a self-consistent gravitational field equation in Finsler spacetime (5).

The notion of Ricci tensor in Finsler geometry was first introduced by Akbar-Zadeh[20]

$$Ric_{\mu\nu} = \frac{\partial^2 \left(\frac{1}{2} F^2 Ric \right)}{\partial y^\mu \partial y^\nu}. \quad (22)$$

And the scalar curvature in Finsler geometry is given as $S = g^{\mu\nu} Ric_{\mu\nu}$. Here, we define the modified Einstein tensor in Finsler spacetime

$$G_{\mu\nu} \equiv Ric_{\mu\nu} - \frac{1}{2} g_{\mu\nu} S. \quad (23)$$

Plugging the equation of Ricci scalar (12) into the formula (23), and noticing that \bar{F} is two dimensional Finsler spacetime with constant flag curvature λ , we obtain

$$G_t^t = \frac{A'}{rA^2} - \frac{1}{r^2A} + \frac{\lambda}{r^2}, \quad (24)$$

$$G_r^r = -\frac{B'}{rAB} - \frac{1}{r^2A} + \frac{\lambda}{r^2}, \quad (25)$$

$$G_\theta^\theta = G_\varphi^\varphi = -\frac{B''}{2AB} - \frac{B'}{2rAB} + \frac{A'}{2rA^2} + \frac{B'}{4AB} \left(\frac{A'}{A} + \frac{B'}{B} \right). \quad (26)$$

Next, we investigate the covariant conserve properties of the tensor G_ν^μ . The covariant derivative of G_ν^μ in Finsler spacetime is given as [4]

$$G_\nu^\mu |_\mu = \frac{\delta}{\delta x^\mu} G_\nu^\mu + \Gamma_{\mu\rho}^\mu G_\nu^\rho - \Gamma_{\mu\nu}^\rho G_\rho^\mu, \quad (27)$$

where

$$\frac{\delta}{\delta x^\mu} = \frac{\partial}{\partial x^\mu} - \frac{\partial G^\rho}{\partial y^\mu} \frac{\partial}{\partial y^\rho}, \quad (28)$$

and $\Gamma_{\mu\rho}^\mu$ is the Chern connection. Here, we have used ‘|’ to denote the covariant derivative. The form of covariant derivative (27) and ‘ δ ’-derivative (28) are well defined such that they transform as tensor under a coordinate change in Finsler spacetime [4]. The Chern connection can be expressed in terms of geodesic spray coefficients G^μ and Cartan connection $A_{\lambda\mu\nu} \equiv \frac{F}{4} \frac{\partial}{\partial y^\lambda} \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y^\nu} (F^2)$

$$\Gamma_{\mu\nu}^\rho = \frac{\partial^2 G^\rho}{\partial y^\mu \partial y^\nu} - A_{\mu\nu|\kappa}^\rho \frac{y^\kappa}{F}. \quad (29)$$

Noticing that the modified Einstein tensor G_ν^μ only depend on r and do not have y -dependence, and Cartan tensor $A_{\mu\nu}^\rho = A_{jk}^i$ (index i, j, k run over θ, φ), one can easily get

that $G_t^\mu{}_{|\mu} = G_\theta^\mu{}_{|\mu} = G_\varphi^\mu{}_{|\mu} = 0$. The proof of $G_r^\mu{}_{|\mu} = 0$ is somewhat subtle. By making use of the equations (8,10,11), we find from (29) and $A_{\mu\nu}^\rho = A_{jk}^i$ that

$$\Gamma_{rt}^t = \frac{B'}{2B}, \quad \Gamma_{r\theta}^\theta = \Gamma_{r\varphi}^\varphi = \frac{1}{r}. \quad (30)$$

Then, after a tedious calculation, one can check that the equation $G_r^\mu{}_{|\mu} = 0$ indeed satisfy. Following the sprite of general relativity, we propose that the gravitational field equation in the given Finsler spacetime (5) should be of the form

$$G_\nu^\mu = 8\pi_F G T_\nu^\mu, \quad (31)$$

where T_ν^μ is the energy-momentum tensor. The volume of Finsler space [21] is generally different with the one of Riemann geometry. We have used $4\pi_F$ to denote the volume of \bar{F} in field equation (31).

For simplicity, we set the energy-momentum tensor to be of the form

$$T_\nu^\mu = \text{diag}(\rho(r), -p(r), -p(r), -p(r)), \quad (32)$$

where $\rho(r)$ and $p(r)$ are the energy density and pressure of the gravitational source, respectively. Then, by making use of equations (24,25,26), we reduce the gravitational field equation to three independent equations

$$\frac{2p'}{\rho + p} = -\frac{B'}{B}, \quad (33)$$

$$\frac{A'}{rA^2} - \frac{1}{r^2A} + \frac{\lambda}{r^2} = 8\pi_F G \rho, \quad (34)$$

$$\frac{B'}{rAB} + \frac{1}{r^2A} - \frac{\lambda}{r^2} = 8\pi_F G p. \quad (35)$$

The solution of equation (34) is given as

$$A^{-1} = \lambda - \frac{2Gm(r)}{r}, \quad (36)$$

where $m(r) \equiv \int_0^r 4\pi_F x^2 \rho(x) dx$. By making use of equation (36), and plugging the equation (35) into (33), we obtain that

$$-r^2 p' = (\rho + p)(4\pi_F G p r^3 + Gm) \left(\lambda - \frac{2Gm}{r} \right)^{-1}. \quad (37)$$

The equation (37) reduces to the famous Oppenheimer-Volkoff equation if Finsler spacetime \bar{F} reduces to two dimensional Riemann sphere. Combining the modified Oppenheimer-Volkoff equation (37) with the equation of state, one can obtain the interior structure of gravitational source.

The interior solution (36) should be consistent with the exterior solution (17) at the boundary of the gravitational source. Therefore, we get

$$a\lambda = 1. \quad (38)$$

At last, combining the boundary condition (38) with the requirement of Newtonian limit (21), we get the exterior solution $B(r)$ and $A(r)$ as

$$B(r) = 1 - \frac{2GM}{\lambda r}, \quad (39)$$

$$A(r) = \left(\lambda - \frac{2GM}{r} \right)^{-1}. \quad (40)$$

IV. THE MOTION OF PARTICLES

Plugging the equations of geodesic spray coefficients (8,9,10,11) into the formula of geodesic equation (2), we obtain the geodesic equation of Finsler spacetime (5)

$$0 = \frac{d^2 t}{d\tau^2} + \frac{B'}{B} \frac{dr}{d\tau} \frac{dt}{d\tau}, \quad (41)$$

$$0 = \frac{d^2 r}{d\tau^2} + \frac{B'}{2A} \left(\frac{dt}{d\tau} \right)^2 + \frac{A'}{2A} \left(\frac{dr}{d\tau} \right)^2 - \frac{r}{A} \bar{F}^2 \left(\frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau} \right), \quad (42)$$

$$0 = \frac{d^2 \theta}{d\tau^2} + \frac{2}{r} \frac{dr}{d\tau} \frac{d\theta}{d\tau} + 2\bar{G}^\theta, \quad (43)$$

$$0 = \frac{d^2 \varphi}{d\tau^2} + \frac{2}{r} \frac{dr}{d\tau} \frac{d\varphi}{d\tau} + 2\bar{G}^\varphi. \quad (44)$$

The solution of equation (41) is

$$B \frac{dt}{d\tau} = 1, \quad (45)$$

where we have set the integral constant to be 1 by the normalization of τ . Noticing that y^μ equals to $\frac{dx^\mu}{d\tau}$ along the geodesic, and by making use of the equations (43,44), we find that

$$\frac{d\bar{F}}{d\tau} = \frac{\partial \bar{F}}{\partial x^i} \frac{dx^i}{d\tau} + \frac{\partial \bar{F}}{\partial y^i} \frac{dy^i}{d\tau} = y^i \left(\frac{\partial \bar{F}}{\partial x^i} - \frac{2\bar{G}_i}{\bar{F}} \right) - 2y^r \bar{F} = -\bar{F} \frac{d \ln r^2}{d\tau}, \quad (46)$$

where $\bar{G}_i = \bar{g}_{ij} \bar{G}^j$ (i, j run over θ, φ), and we have used the fact that \bar{F} is a homogenous function of y of degree 1 to derive the third equation of (46). The solution of equations (46) is given as

$$r^2 \bar{F} = J, \quad (47)$$

where J is an integral constant. By making use of the equations (45,47), we find from geodesic equation (42) that

$$\frac{d}{d\tau} \left(A \left(\frac{dr}{d\tau} \right)^2 + \frac{J^2}{r^2} - \frac{1}{B} \right) = 0. \quad (48)$$

The solution of (48) is given as

$$A \left(\frac{dr}{d\tau} \right)^2 + \frac{J^2}{r^2} - \frac{1}{B} = A \left(\frac{dr}{d\tau} \right)^2 + r^2 \bar{F}^2 - B \left(\frac{dt}{d\tau} \right)^2 = -F^2, \quad (49)$$

where we have used the equations (45,47) to derive the second equation of (49). The equation (49) means that F is constant along the geodesic.

Now, we have three solutions (45,47,49) of the geodesic equations, the fourth one depends on the explicit form of two dimension Finsler space \bar{F} . However, we can still find some information of particle motion from the obtained solutions. Consider a particle move along radial direction, combining the equation (45) with (47), and by making use of the exterior solutions (39,40) of $B(r)$ and $A(r)$, we obtain that

$$\frac{dr}{dt} = \sqrt{\lambda^{-1} - F^2 \left(1 - \frac{2GM}{\lambda r} \right) \left(1 - \frac{2GM}{\lambda r} \right)}. \quad (50)$$

It is obvious from (50) that $\frac{dr}{dt} \rightarrow 0$ while $r \rightarrow 2GM/\lambda$. The modified Schwarzschild radius in Finsler spacetime (5) is

$$r_s = \frac{2GM}{\lambda}. \quad (51)$$

V. TWO DIMENSIONAL FINSLER SPACE WITH CONSTANT FLAG CURVATURE

In this section, we will present an example of two dimensional Finsler space with constant positive flag curvature.

Bao *et al.* [22] have given a completely classification of Randers-Finsler space [23] with constant flag curvature. A two dimensional Randers-Finsler space with constant positive flag curvature $\lambda = 1$ is given as

$$\bar{F} = \frac{\sqrt{(1 - \epsilon^2 \sin^2 \theta) y^\theta y^\theta + \sin^2 \theta y^\varphi y^\varphi}}{1 - \epsilon^2 \sin^2 \theta} - \frac{\epsilon \sin^2 \theta y^\varphi}{1 - \epsilon^2 \sin^2 \theta}, \quad (52)$$

where $0 \leq \epsilon < 1$. It is obvious that the metric (52) returns to Riemannian sphere while $\epsilon = 0$. And the metric (52) is non-reversible for $\varphi \rightarrow -\varphi$. The Randers-Finsler space (52) has two

close geometrically distinct closed geodesics [24] if ϵ is irrational. The two geodesics locate at $\theta = \frac{\pi}{2}$ with length $L_{\pm} = 2\pi(1 \pm \epsilon)^{-1}$. This fact can be proved by plugging the metric (52) into the formula of geodesic equation (2). Then, one can find that $\theta = \frac{\pi}{2}$ and $\varphi = u\tau + v$ (u, v are integral constants) are the solutions of geodesic equation. The Randers-Finsler space (52) is homotopy equivalent to the two dimensional sphere [24].

In terms of Busemann-Hausdorff volumm form, the volume of a close Randers-Finsler surface $F = \sqrt{a_{ij}(x)y^i y^j} + b_i(x)y^i$ is given as [21]

$$Vol_F = \int (1 - (a^{ij}b_i b_j))^{\frac{3}{2}} \sqrt{\det(a_{ij})} dx^1 \wedge dx^2. \quad (53)$$

Plugging the Randers metric (52) into the formula (53), we obtain that its volume is 4π , which is the same with unit Riemannian sphere.

VI. CONCLUSIONS AND REMARKS

In view of geodesic deviation equation, the vacuum field equation $Ric = 0$ in Finsler spacetime implies that the geodesic rays are parallel to each other. The geometric invariant property of Ricci scalar implies that the vacuum field equation is insensitive to the connection, which is an essential physical requirement. Starting from the ansatz (5), we have found an exact solution of vacuum field equation (16,17).

A general gravitational field equation in Finsler spacetime is still to be completed. However, we have found that the gravitational field equation for the metric (5) should be the form as (31). And, we have proved that the Finslerian covariant derivative of the geometrical part of the gravitational field equation is conserved. It is obvious that the gravitational field equation (31) returns to the vacuum field equation while the energy-momentum tensor vanishes. We have found an interior solution of the gravitational field equation (31). The interior solution (36) consistent with the exterior solution (17) at the boundary of the gravitational source. And we required that the exterior solution should return to Newton's gravity. The two boundary conditions constrain that the exterior solution should be the form as (39,40). Together with specific form of two dimensional Finsler space (52) with constant

positive flag curvature $\lambda = 1$, the exterior metric of vacuum field solution was given as

$$F^2 = \left(1 - \frac{2GM}{r}\right) y^t y^t - \left(1 - \frac{2GM}{r}\right)^{-1} y^r y^r - r^2 \left(\frac{\sqrt{(1 - \epsilon^2 \sin^2 \theta) y^\theta y^\theta + \sin^2 \theta y^\varphi y^\varphi} - \epsilon \sin^2 \theta y^\varphi}{1 - \epsilon^2 \sin^2 \theta} \right)^2. \quad (54)$$

The metric (54) is no other than the Schwarzschild metric except for the change from Riemannian sphere to “Finslerian sphere” (52). We have presented three solutions (45,47,49) of geodesic equations of the metric (54). The fourth one depends on the geodesic equation of the “Finslerian sphere” (52). The geometrical properties of ‘Finslerian sphere’ (52) are as follows. It is non-reversible for $\varphi \rightarrow -\varphi$, it has two close geodesics locate at $\theta = \frac{\pi}{2}$ with length $L_\pm = 2\pi(1 \pm \epsilon)^{-1}$, its volume that is the surface volume of unit “Finslerian sphere” equals to 4π , it only have one independent Killing vector $V^\varphi = \text{constant}$.

The Schwarzschild spacetime will return to Minkowski spacetime if there is no gravitational source. As for the Finslerian vacuum spacetime (54), if there is no gravitational source, namely, $M = 0$, the metric reduces to

$$F^2 = y^t y^t - y^r y^r - r^2 \left(\frac{\sqrt{(1 - \epsilon^2 \sin^2 \theta) y^\theta y^\theta + \sin^2 \theta y^\varphi y^\varphi} - \epsilon \sin^2 \theta y^\varphi}{1 - \epsilon^2 \sin^2 \theta} \right)^2. \quad (55)$$

According to the formula (12), the Ricci scalar or Ricci tensor of the metric (55) equal to 0. This fact holds even for three dimensional subspace of the metric (55). However, the metric (55) or its spatial part is not a flat Finsler spacetime. The Finsler spacetime is a flat one [4] if and only if $Ric = 0$ and the geodesic spray coefficients G^μ is quadratic in y . This fact is quite different with Riemann geometry. It is well known that three dimensional Riemann space is flat while its Ricci tensor equal to 0. Nevertheless, even the space part of the metric (55) is not a flat Finsler space.

In Riemann spacetime without torsion, at any fixed point, one can erect a local coordinate system such that the metric is Minkowskian. One necessary condition of this statement is the Riemann metric is quadric. However, this necessary condition does not hold in a general Finsler metric. Therefore, Finsler spacetime is not locally isometric to a Minkowski spacetime. One deduction of it is the speed of light is not locally isotropic. The propagation of light obeys $F = 0$. One can find from the local metric that the radial speed of light equals to 1. And the non-radial speed of light satisfy

$$c_\theta^2 + (c_\varphi - \epsilon \sin \theta)^2 = 1, \quad (56)$$

where $c_\theta \equiv \frac{d\theta}{dt}$ and $c_\varphi \equiv \frac{d\varphi}{dt} \sin \theta$.

The Schwarzschild radius form an event horizon in Schwarzschild spacetime. The Schwarzschild solution can be maximally extended by Kruskal extension. The coordinate transformation between Schwarzschild metric and Kruskal metric is only related to r and t . Therefore, one can also get a maximally extended Finslerian vacuum solution (54) by Kruskal's method.

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